

THE UNIVERSITY OF CHICAGO

DYNAMICS OF VARIABLE MASSES

A PART OF A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1934

by

EVAN JOHNSON II

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VITA

Evan Johnson, Jr., was born on August 15, 1910, in Oak Park, Illinois. There he received his grammar school and secondary school training, being graduated from the Oak Park and River Forest Township High School in 1926. In the autumn of that year he entered the University of Chicago. He received the degree of Bachelor of Science in 1930. He was appointed Assistant in Mathematics at the Pennsylvania State College for the academic years 1930-1933. During that time he took courses under Professors Owens, Curry, Frink, Rupp and Sheffer, receiving the degree of Master of Arts in 1932. He attended the graduate school of the University of Chicago during the spring quarter of 1930, the summer of 1931 and the five consecutive quarters beginning with the summer of 1933. It was his privilege to attend lectures of most of the members of the Faculty of Mathematics. To them all he is indebted. To Professor Bartky he wishes to express, in addition, his appreciation of the generous assistance which has made possible the preparation of this paper.

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DYNAMICS OF VARIABLE MASSES

By EVAN JOHNSON II

(PART I)

INTRODUCTION

The theory that stellar energy arises from the annihilation of matter was advanced by both MACMILLAN* and JEANS.** Under such a theory it becomes necessary to consider mass as a function of time in the treatment of dynamical problems.

In the study of the motions of particles of variable mass the second law of motion has been chosen, usually, from one of the two forms "force equals mass times acceleration" and "force equals rate of change of momentum." To only one of these equations can the interpretations of ordinary Newtonian mechanics be applied. In the present paper it is proposed to investigate postulates and definitions upon which to base systems of dynamics employing each of these laws. It will be seen that a similar discussion can be given for still more general laws of motion.

It is to be noted that the title "Variable Masses" has been applied to two or three different problems. A star gathering in matter from nebulosity may be treated mathematically as if it were a body whose mass is an increasing function of the time. Again, if an astronomical body is passing through an atmospheric medium, the heat of friction may vaporize a part of the solid mass. Mathematically, we may treat such a body as one of decreasing mass. Both of these problems, however, are purely Newtonian, since the law of conservation of mass is employed in arriving at the equations of motion. On the other hand, the theory of annihilation of matter clearly requires that the principle of conservation be abandoned. It is assumed that matter is transformed into energy. A list of papers treating these problems is given in the bibliography.

(PART II)

ANALYSIS OF THE LAWS OF MOTION

The physical quantities, length, mass, and time, are the subject matter of dynamics. Postulates defining the measurements of these quantities and specifying their relationships provide the basis for a particular system of dynamics.

The concept of fixed space seems untenable in any dynamical system. It is apparently necessary to consider at least two particles in space, relating measurements to an observer fixed on one of them. We shall take as undefined the measurement of length and, also, the notion of particles moving independently of each other. These particles we shall refer to as "freely moving" or "free" particles.

The measurements of time, mass, and force are made to depend upon the measurement of length by means of the laws of motion.

1. *Invariance under Transformation of Axes.* We shall consider a group of free particles, each of which may be selected as the origin of a system of rectangular coordinates. For convenience, we may choose corresponding axes to be parallel. Methods of measurement, defined by postulates, must apply in any of these coordinate systems.

The first law prescribes the relative motion of two free particles, m_1 and m_2 , i. e., a transformation between the coordinate systems of two free observers. In the statement of any general law of motion only those equations are admissible which are invariant under this transformation. Otherwise, we should have to make a selection of particular axes for the application of the law.

2. *A First Law of Motion.* Let us suppose that the motion of a free particle of mass m , relative to a free

*) Numbered references in footnotes apply to items in the bibliography at the end of this paper. 9, 11, 12, 13. **) 3, 5, 6.

observer, is represented by the homogeneous linear equation

$$(1) \quad amr'' + bm'r' + cm''r = 0,$$

where r is the vector distance to the particle and the prime indicates a derivative with respect to the time t . If c and b are constants, not both zero, and if $m=m(t)$, it may be shown that the only transformation of the form $r=s - f(t)$, independent of m , under which equation (1) is invariant is $r - s = \text{constant vector}$. If $c = b=0$, equation (1) becomes $r''=0$, invariant, of course, under the transformation $r=s - kt$, where k is a constant vector. The only equation of the form (1), therefore, which is invariant for moving systems of coordinates is $r''=0$. If we then restrict ourselves to equations of type (1) in the statement of the first law, we must assume $r''=0$. We shall state the first law of motion in the following form, to avoid the use of the term force, which depends for its measurement upon the second law: "Equal intervals of time are measured by equal intervals of separation of two free bodies in space."

An observer's selection of a free particle for the measurement of time is entirely arbitrary. The assumption that any two free particles may be chosen for the measurement of the same parameter is an assumption concerning physical phenomena.

It is of interest to note JEANS' article in the *Monthly Notices*.* The arguments used in that paper substantiate the conclusion above regarding the first law of motion. In view of the further arguments of the present paper, however, we do not believe that they prove the point which he wishes to make.

3. *The Second and Third Laws of Motion.* The measurements of the quantities mass and force are defined by the second law of motion, upon application of the third law. There are several possible choices for these laws, each carrying with it a definition of and an equation for the measurement of force. In any case, by section 2, the equation of motion which is used must be invariant under the transformation $r=s - kt$, where k is a constant.

We shall consider first the case where the force F is defined by the equation $F=mr''$. If we assume that the third law of motion shall be adopted in its usual form, we may write the equations of a two-particle system

$$(2) \quad \begin{aligned} m_1 r_1'' &= -F_{12} \\ m_2 r_2'' &= F_{12}. \end{aligned}$$

On adding these two equations we obtain the relation

$$(3) \quad m_1 r_1'' + m_2 r_2'' = 0.$$

Equation (3) gives us the following interpretation of

*) 4.

the second law of motion: The mass ratio of two particles is defined to be the ratio of the accelerations which the particles exhibit to a free observer when moving only under their mutual influence. We assume that the mass ratio, determined in this manner, is an invariant property of the pair, i. e., independent of the coordinate system in which it is measured. Upon selection of an arbitrary standard we may associate with every particle a function $m(t)$ called its mass.

Since the quantity mr'' is invariant under linear transformation, it is apparent that all free observers will agree in their measurements of the force acting upon a given particle.

The quantity $(mr')'$ is not invariant under the linear transformation $r=s - kt$. Consequently, if we define force by the equation $F=(mr')'$, the measurement either of force or of mass must differ with each observer. Let us formulate definitions upon the assumption that all free observers agree in the measurement of mass.

Suppose that $F=(mr')'$. In order to arrive at an equation to define the measurement of mass we must first specify some further property of the quantity F . To do this we shall select a third law of motion.

In classical mechanics it is the third law which makes possible the representation of the motion of a closed system of particles by means of the motion of a single hypothetical particle. In the present case we shall attempt to give the same interpretation to the third law. A free observer describes the motions of two free particles, m_1 and m_2 , by the equations

$$(4) \quad r_1' = k_1 \text{ and } r_2' = k_2.$$

The equation of momentum for the pair of particles takes the form

$$(5) \quad m_1 r_1' + m_2 r_2' = m_1 k_1 + m_2 k_2.$$

The equations of motion of a two-particle system, moving under only its own gravitational attraction, should, presumably, give the same integral of momentum. Let the equations of motion of the system be written

$$(6) \quad (m_1 r_1')' = -F_{12} \text{ and } (m_2 r_2')' = -F_{21},$$

and let us assume that upon addition of the two equations we obtain

$$(7) \quad (m_1 r_1')' + (m_2 r_2')' = m_1' k_1 + m_2' k_2,$$

where k_1 and k_2 are the initial velocities. Upon integration we obtain the desired equation (5), since the constant must be set equal to zero. We may state the third law of motion for the general case as follows: Forces acting within a closed system of particles are paired so that the momentum of the system is the same as if the particles were moving freely. Upon these assumptions we may formulate the procedure for the measurement of the quantities involved in the equations.

Let a free observer be located at the origin of a coordinate system. If two particles are moving under only their own gravitational influence the ratio m_2/m_1 of their masses shall be defined to be a function of the time t satisfying equation (5). We shall assume that all free observers agree upon the determination of the mass ratio.

The force F , acting upon any particle of mass $m(t)$, shall be defined as $(mr')'$. We may now determine certain properties of the function F . We shall find these properties differing considerably from those intuitively associated with force in classical mechanics.

We observe that the measurement of force is dependent upon the coordinate system of the observer. Let the force F acting upon a particle m_1 be measured by observers 1 and 2 and be designated by F_{11} and F_{12} , respectively. From the first law of motion, these two quantities must be connected by the relation

$$(8) \quad F_{12} = F_{11} - V_{21}m_1',$$

where V_{21} is the velocity of observer 2, measured by observer 1.

Again contrary to the Newtonian case, it will be seen that a freely moving particle does not move under zero force. A free observer concludes from the equation

$$(9) \quad F = (mr')' = (mk)' = m'k$$

that a free particle of mass m , moving with constant velocity k , is acted upon by the force $m'k$. This result is consistent with equation (8) above.

Making use of the above assumptions, we shall now show that it is possible to determine a formula for gravitational attraction expressible in any reference system and implying the Newtonian law for constant masses.

(PART III) GRAVITATION

In the previous section we have formulated postulates for two dynamical systems. The selection of a law of gravitation in each of these systems must be governed by consideration of the properties of the corresponding force functions.

1. *The Selection of a Law.* Using the law of motion $F=mr''$, we have found that the properties of the force function are essentially those of classical dynamics. Therefore, we select the usual form of the law of gravitation.

The definition of force $F=(mr')'$ has¹ been seen to require a transformation relating the measurements of force by two different free observers. Consequently, it is necessary in this case that the expression for the gravitational attraction between two particles be invariant under that transformation. In the determination of such a law we shall be guided by two further assumptions:

(1) The formula for attraction shall imply, as a special case, the Newtonian law for constant masses.

(2) The equations of motion of a system of n particles relative to any one of them shall be independent of translation.

Let the equations of the n -body problem be written

$$(10) \quad (m_i r_i')' = -k^2 m_i \sum_{j \neq i} m_j \frac{r_i - r_j}{r_{ij}^3} - g_i^{(n)}, \quad (i=1, 2, \dots, n),$$

where the function $g_i^{(n)}$ is to be determined in accordance with assumptions (1) and (2) above. Upon addition of the n equations of (10) we obtain

$$(11) \quad \Sigma (m_i r_i')' = - \Sigma g_i^{(n)}.$$

The third law of motion requires the equation

$$(12) \quad \Sigma (m_i r_i')' = \Sigma m_i' k_i \equiv G'.$$

From equations (11) and (12) it follows that

$$(13) \quad \Sigma g_i^{(n)} = -G'.$$

Let the particle m_j be selected as the origin of a new system of coordinates, designating by r_{ij} the vector to the particle m_i . From the previous equations we can show that

$$(14) \quad G - \Sigma m_i r_{ij}' = r_j' \Sigma m_i = M r_j'.$$

Upon solving this equation for r_j' and substituting in the j th equation of (10) we obtain a relation which must be independent of G (assumption 2). Therefore,

$$(15) \quad g_j^{(n)} = - \left(\frac{m_j G}{M} \right)', \quad (j = 1, 2, \dots, n).$$

We observe that the values of $g_i^{(n)}$ satisfy equation (13) in accordance with the third law of motion. Also, when the masses are constant each of the g 's separately vanish, implying the Newtonian law. Finally, we note that the functions $g_i^{(n)}$ are formally invariant under the transformation (8), above.

2. *Discussion of the Gravitational Theory.* To express the attractions between two particles we have added to the Newtonian formula functions $-g_i^{(2)}$, computed to satisfy prescribed conditions. In the case of n particles we have added functions $-g_i^{(n)}$. The functions $g_i^{(n)}$ imply the $g_i^{(2)}$ as a special case but are not expressible as combinations of them. The equation for $g_i^{(n)}$ may be written in the form

$$(16) \quad -g_i^{(n)} - m_i' k_i = \sum_{j=1}^n \left(\frac{m_i m_j}{M} \right)' (k_j - k_i), \quad (i=1, 2, \dots, n),$$

which exhibits the pairing of forces required by the third law of motion.

It is desirable to assume that the equations of motion of particles moving under gravitational attraction apply only in a finite region. If we allow the distances between particles to become infinite, the equations (10), for example, still involve the velocities, which are indeterminate.

In each of the dynamical systems under discussion the solution of the two-body problem can be found without difficulty. The formulas for the period and eccentricity suggest that certain distributions should be found among the elements of double stars. The computed relations can be checked against observation.

1. *The Problem of Two Bodies.* The problem of two bodies using the law $F=mr''$ has been treated by several writers.* They agree apparently on the following results: The major axis varies inversely as the sum of the masses and the period inversely as the square of the sum of the masses; the eccentricity remains invariant.

Previous treatments of the two-body problem using the law $F=(mr')'$ have differed considerably. In general, the Newtonian form of the gravitational formula has been adopted. The fact that the equations of motion in this form are not invariant under linear transformation has given rise to some doubtful interpretations.

Following the discussion of Part III we may write the equations for the two-body problem

$$(17) \quad (m_i r_i')' = -k^2 \frac{m_1 m_2}{r_{12}^3} (r_i - r_j) - g_i^{(2)}, (i=1, 2; j \neq i),$$

and from these equations we obtain the equation of relative motion

$$(18) \quad \left(\frac{m_1 m_2}{m_1 + m_2} r_{12}' \right)' = \frac{-k^2 m_1 m_2}{r_{12}^3} r_{12}.$$

After crossing equation (18) with r_{12} and integrating, we obtain

$$(19) \quad \frac{m_1 m_2}{m_1 + m_2} r_{12}' \times r_{12} = h.$$

Equation (19) shows that the motion is in a plane.

After transformation to polar coordinates the equation of the orbit can be found in the usual way by the substitution of $1/u$ for r . The final solution can be accomplished by the method of variation of parameters. Considering only the secular variation of the elements, it is found that

$$(20) \quad \theta_0 = \text{constant}; e = \text{constant}; a = \alpha/M^3 \sigma^2; P = \beta/M^5 \sigma^3, \text{ where } \alpha \text{ and } \beta \text{ are constants, } M = m_1 + m_2, \text{ and } \sigma = m_1 m_2 / M^2; \sigma \text{ depends only on the mass ratio.}$$

2. *Statistical Results.* Statistical discussion of binaries has centered around two types of distribution.** Following the arguments of an earlier paper by OTTO STRUVE,† MARKOWITZ has shown that, if the mass ratio is assumed to be constant, it is possible to determine statistical relations among spectroscopic binaries

connecting the period and the semi-amplitude of the velocity of the more massive component. Correlation may also be found between the periods and the eccentricities of the orbits. We shall be interested principally in the first problem.

Let the average of a set of elements a_i be designated by $\bar{a}$. It can be shown that, if the maximum difference of any two of the a 's is sufficiently small, compared to the a 's themselves, $|\bar{a}^n - \overline{a^n}|$ and $|\bar{a}^{1/n} - \overline{a^{1/n}}|$ are negligible. Likewise, $|\bar{a} \bar{b} - \overline{ab}|$ is negligible if the maximum difference of the elements of either set is small.

The following is a known relation for spectroscopic binaries:

$$(21) \quad K P^{1/3} = \frac{c C M^{1/3} \sin i}{(1 + e) \sqrt{1 - e^2}},$$

where K is the semi-amplitude of the velocity of the more massive component, C a constant, and e the mass ratio. If we assume that the mass ratio is constant, we may write equation (21) in the form

$$(22) \quad K P^{1/3} = M^{1/3} C_1.$$

From the argument of the preceding paragraph, equation (22) may be written in the form

$$(23) \quad \bar{K} \bar{P}^{1/3} = \bar{M}^{1/3} C_1,$$

for purposes of statistical discussion, if the averages are taken over classes of doubles with sufficiently small differences of period. Finally, from equation (23) we obtain

$$(24) \quad \bar{K} \bar{P}^{2/5} = D_1 \text{ if } \bar{M} \bar{P}^{1/5} = D \text{ and}$$

$$(25) \quad \bar{K} \bar{P}^{1/2} = D_2 \text{ if } \bar{M} \bar{P}^{1/2} = D.$$

The period of a double picking up mass from nebulousity is given by the formula $P = a_1/M^5$.* Upon the assumption that $F = (mr')'$ when mass is being reduced by radiation we have found the equivalent formula $P = a_2/M^5$. Therefore, the two variations of mass occurring together require the relation (24), $\bar{K} \bar{P}^{2/5} = D_1$, to hold among groups of doubles with small differences of period.

Upon the assumption that $F = mr''$ for radiation, the period is given by the formula $P = a/M^2$. Radiation alone, therefore, would require the relation (25), and the picking up of matter, without radiation, the relation (24). Since the process of radiation is essentially continuous, while the gathering of mass occurs only at intervals, we may assume equation (25), $\bar{K} \bar{P}^{1/2} = D_2$, as a consequence of both variations together.

The accompanying table (page 165) gives the values of D_1 and D_2 , computed from observation for certain classes of doubles.** This group has been selected, since the relative number of binaries having longer or shorter periods is considerably less.

*) 1, 3 page 291, 4, 10.

**) 3, 14, 18.

†) 22.

*) 14, 15.

**) 14.

THE PERIOD AMPLITUDE FUNCTIONS

Group	Log 10 P	$\bar{P}$	$\bar{K}$	n*	$D_1 = \bar{K}\bar{P}^{2/5}$	$D_2 = \bar{K}\bar{P}^{1/2}$
1.....	1.20	1.267	90.0	18	98.94	101.3
2.....	1.50	2.362	84.4	45	119.0	129.7
3.....	1.80	4.391	61.4	55	111.0	128.7
4.....	2.10	9.430	52.2	48	128.1	160.3
5.....	2.40	17.77	51.4	28	162.5	216.7
6.....	2.70	34.71	36.2	20	149.6	213.3
7.....	3.00	67.15	29.5	15	158.7	241.7

*) n indicates the number of stars in each group.

The values of D_1 show greater uniformity than those of D_2 . If we accept the values of D_1 as being constant, we conclude that observation tends to confirm the law $F = (mr')'$ rather than the law $F = mr''$.

Under each of the laws of motion we have found that the eccentricity remains invariant. The period-eccentricity distribution, therefore, will be essentially the same in each case. The changes in eccentricity due to the picking up of matter have been discussed by MARKOWITZ.*

*) 14, 15.

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